NOTE ON MMAT 5010: LINEAR ANALYSIS (2017 1ST TERM)

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1. Normed spaces

Throughout this note, we always denote $\mathbb{K}$ by the real field $\mathbb{R}$ or the complex field $\mathbb{C}$. Let $\mathbb{N}$ be the set of all natural numbers. Also, we write a sequence of numbers as a function $x : \{1, 2, ...\} \to \mathbb{K}$.

Definition 1.1. *Let X be a vector space over the field $\mathbb{K}$. A function $\|\cdot\| : X \to \mathbb{R}$ is called a norm on X if it satisfies the following conditions.*

(i) $\|x\| \geq 0$ *for all* $x \in X$ *and* $\|x\| = 0$ *if and only if* $x = 0$.
(ii) $\|\alpha x\| = |\alpha|\|x\|$ *for all* $\alpha \in \mathbb{K}$ *and* $x \in X$.
(iii) $\|x + y\| \leq \|x\| + \|y\|$ *for all* $x, y \in X$.

In this case, the pair $(X, \|\cdot\|)$ is called a normed space.
Also, the distance between the elements x and y in X is defined by $\|x - y\|$.

The following examples are important classes in the study of functional analysis.

Example 1.2. *Consider $X = \mathbb{K}^n$. Put*

$$\|x\|_p := \big(\sum_{i=1}^{n} |x_i|^p\big)^{1/p} \quad \text{and} \quad \|x\|_\infty := \max_{i=1,\dots,n} |x_i|$$

for $1 \leq p < \infty$ and $x = (x_1, ..., x_n) \in \mathbb{K}^n$.
Then $\|\cdot\|_p$ (called the usual norm as p=2) and $\|\cdot\|_\infty$ (called the sup*-norm) all are norms on $\mathbb{K}^n$.*

Example 1.3. *Put*

$$c_0 := \{(x(i)) : x(i) \in \mathbb{K},\ \lim |x(i)| = 0\} \text{(called the null sequnce space)}$$

and

$$\ell^\infty := \{(x(i)) : x(i) \in \mathbb{K},\ \sup_i |x(i)| < \infty\}.$$

Then c_0 is a subspace of ℓ^∞. The sup*-norm $\|\cdot\|_\infty$ on ℓ^∞ is defined by*

$$\|x\|_\infty := \sup_i |x(i)|$$

for $x \in \ell^\infty$. Let

$$c_{00} := \{(x(i)) : \text{ there are only finitly many } x(i)\text{'s are non-zero}\}.$$

Also, c_{00} is endowed with the sup*-norm defined above is called the finite sequence space.*

Example 1.4. *For $1 \leq p < \infty$, put*

$$\ell^p := \{(x(i)) : x(i) \in \mathbb{K},\ \sum_{i=1}^{\infty} |x(i)|^p < \infty\}.$$

Also, ℓ^p is equipped with the norm

$$\|x\|_p := (\sum_{i=1}^{\infty} |x(i)|^p)^{\frac{1}{p}}$$

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for $x \in \ell^p$. Then $\|\cdot\|_p$ is a norm on ℓ^p (see [1, Section 9.1]*).*

Example 1.5. *Let $C^b(\mathbb{R})$ be the space of all bounded continuous $\mathbb{R}$-valued functions f on $\mathbb{R}$. Now $C^b(\mathbb{R})$ is endowed with the* sup*-norm, that is,*

$$\|f\|_\infty = \sup_{x\in\mathbb{R}} |f(x)|$$

for every $f \in C^b(\mathbb{R})$. Then $\|\cdot\|_\infty$ is a norm on $C^b(\mathbb{R})$.
Also, we consider the following subspaces of $C^b(X)$.
Let $C_0(\mathbb{R})$ $\big($resp. $C_c(\mathbb{R})\big)$ be the space of all continuous $\mathbb{R}$-valued functions f on $\mathbb{R}$ which vanish at infinity (resp. have compact supports), that is, for every $\varepsilon > 0$, there is a $K > 0$ such that $|f(x)| < \varepsilon$ (resp. $f(x) \equiv 0$) for all $|x| > K$.
It is clear that we have $C_c(\mathbb{R}) \subseteq C_0(\mathbb{R}) \subseteq C^b(\mathbb{R})$.
Now $C_0(\mathbb{R})$ and $C_c(\mathbb{R})$ are endowed with the sup*-norm $\|\cdot\|_\infty$.*

Notation 1.6. *From now on, $(X, \|\cdot\|)$ always denotes a normed space over a field $\mathbb{K}$.*
For $r > 0$ and $x \in X$, let

- *(i) $B(x,r) := \{y \in X : \|x-y\| < r\}$ (called an open ball with the center at x of radius r) and $B^*(x,r) := \{y \in X : 0 < \|x-y\| < r\}$*
- *(ii) $B(x,r) := \{y \in X : \|x-y\| \le r\}$ (called a closed ball with the center at x of radius r).*

Put $B_X := \{x \in X : \|x\| \le 1\}$ and $S_X := \{x \in X : \|x\| = 1\}$ the closed unit ball and the unit sphere of X respectively.

Definition 1.7. *Let A be a subset of X.*

- *(i) A point $a \in A$ is called an interior point of A if there is $r > 0$ such that $B(a,r) \subseteq A$. Write $int(A)$ for the set of all interior points of A.*
- *(ii) A is called an open subset of X if $int(A) = A$.*

Example 1.8. *We keep the notation as above.*

- *(i) Let $\mathbb{Z}$ and $\mathbb{Q}$ denote the set of all integers and rational numbers respectively If $\mathbb{Z}$ and $\mathbb{Q}$ both are viewed as the subsets of $\mathbb{R}$, then $int(\mathbb{Z})$ and $int(\mathbb{Q})$ both are empty.*
- *(ii) The open interval $(0,1)$ is an open subset of $\mathbb{R}$ but it is not an open subset of $\mathbb{R}^2$. In fact, $int(0,1) = (0,1)$ if $(0,1)$ is considered as a subset of $\mathbb{R}$ but $int(0,1) = \emptyset$ while $(0,1)$ is viewed as a subset of $\mathbb{R}^2$.*
- *(iii) Every open ball is an open subset of X (***Check!!***).*

Definition 1.9. *We say that a sequence (x_n) in X converges to an element $a \in X$ if $\lim \|x_n - a\| = 0$, that is, for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $\|x_n - a\| < \varepsilon$ for all $n \ge N$.*
In this case, (x_n) is said to be convergent and a is called a limit of the sequence (x_n).

Remark 1.10.

(i) *If (x_n) is a convergence sequence in X, then its limit is unique. In fact, if a and b both are the limits of (x_n), then we have $\|a-b\| \le \|a-x_n\| + \|x_n - b\| \to 0$. So, $\|a-b\| = 0$ which implies that $a = b$.*
From now on, we write $\lim x_n$ for the limit of (x_n) provided the limit exists.

(ii) *The definition of a convergent sequence (x_n) depends on the underling space where the sequence (x_n) sits in. For example, for each $n = 1, 2...$, let $x_n(i) := 1/i$ as $1 \le i \le n$ and $x_n(i) = 0$ as $i > n$. Then (x_n) is a convergent sequence in ℓ^∞ but it is not convergent in c_{00}.*

Definition 1.11. *Let A be a subset of X.*

(i) *A point $z \in X$ is called a limit point of A if for any $\varepsilon > 0$, there is an element $a \in A$ such that $0 < \|z - a\| < \varepsilon$, that is, $B^*(z, \varepsilon) \cap A \neq \emptyset$ for all $\varepsilon > 0$.*
Furthermore, if A contains the set of all its limit points, then A is said to be closed in X.

(ii) *The closure of A, write $\overline{A}$, is defined by*

$$\overline{A} := A \cup \{z \in X : z \text{ is a limit point of } A\}.$$

Remark 1.12. *With the notation as above, it is clear that a point $z \in \overline{A}$ if and only if $B(z, r) \cap A \neq \emptyset$ for all $r > 0$. This is also equivalent to saying that there is a sequence (x_n) in A such that $x_n \to a$. In fact, this can be shown by considering $r = \frac{1}{n}$ for $n = 1, 2....$*

Proposition 1.13. *With the notation as before, we have the following assertions.*

(i) *A is closed in X if and only if its complement $X \setminus A$ is open in X.*

(ii) *The closure $\overline{A}$ is the smallest closed subset of X containing A. The "smallest" in here means that if F is a closed subset containing A, then $\overline{A} \subseteq F$.*
Consequently, A is closed if and only if $\overline{A} = A$.

Proof. If A is empty, then the assertions (i) and (ii) both are obvious. Now assume that $A \neq \emptyset$. For part (i), let $C = X \setminus A$ and $b \in C$. Suppose that A is closed in X. If there exists an element $b \in C \setminus int(C)$, then $B(b, r) \subsetneq C$ for all $r > 0$. This implies that $B(b, r) \cap A \neq \emptyset$ for all $r > 0$ and hence, b is a limit point of A since $b \notin A$. It contradicts to the closeness of A. So, $A = int(A)$ and thus, A is open.
For the converse of (i), assume that C is open in X. Assume that A has a limit point z but $z \notin A$. Since $z \notin A$, $z \in C = int(C)$ because C is open. Hence, we can find $r > 0$ such that $B(z, r) \subseteq C$. This gives $B(z, r) \cap A = \emptyset$. This contradicts to the assumption of z being a limit point of A. So, A must contain all of its limit points and hence, it is closed.

For part (ii), we first claim that $\overline{A}$ is closed. Let z be a limit point of $\overline{A}$. Let $r > 0$. Then there is $w \in B^*(z, r) \cap \overline{A}$. Choose $0 < r_1 < r$ small enough such that $B(w, r_1) \subseteq B^*(z, r)$. Since w is a limit point of A, we have $\emptyset \neq B^*(w, r_1) \cap A \subseteq B^*(z, r) \cap A$. So, z is a limit point of A. Thus, $z \in \overline{A}$ as required. This implies that $\overline{A}$ is closed.
It is clear that $\overline{A}$ is the smallest closed set containing A.
The last assertion follows from the minimality of the closed sets containing A immediately.
The proof is finished. □

Example 1.14. *Retains all notation as above. We have $\overline{c_{00}} = c_0 \subseteq \ell^\infty$.*
Consequently, c_0 is a closed subspace of ℓ^∞ but c_{00} is not.

Proof. We first claim that $\overline{c_{00}} \subseteq c_0$. Let $z \in \ell^\infty$. It suffices to show that if $z \in \overline{c_{00}}$, then $z \in c_0$, that is, $\lim_{i \to \infty} z(i) = 0$. Let $\varepsilon > 0$. Then there is $x \in B(z, \varepsilon) \cap c_{00}$ and hence, we have $|x(i) - z(i)| < \varepsilon$ for all $i = 1, 2.....$ Since $x \in c_{00}$, there is $i_0 \in \mathbb{N}$ such that $x(i) = 0$ for all $i \geq i_0$. Therefore, we have $|z(i)| = |z(i) - x(i)| < \varepsilon$ for all $i \geq i_0$. So, $z \in c_0$ as desired.

For the reverse inclusion, let $w \in c_0$. It needs to show that $B(w, r) \cap c_{00} \neq \emptyset$ for all $r > 0$. Let $r > 0$. Since $w \in c_0$, there is i_0 such that $|w(i)| < r$ for all $i \geq i_0$. If we let $x(i) = w(i)$ for $1 \leq i < i_0$ and $x(i) = 0$ for $i \geq i_0$, then $x \in c_{00}$ and $\|x - w\|_\infty := \sup_{i=1,2...} |x(i) - w(i)| < r$ as required. □

References

[1] J. Muscat, Functional Analysis, Springer, (2014).
[2] H. Royden and P. Fitzpatrick, Real analysis, fourth edition, Pearson, (2010).